A THREE-DIMENSIONAL PROBLEM OF DIFFRACTION OF AN ELASTIC WAVE AT A SHARP EDGE

(PROSTRANSTVENNAIA ZADACHA DIFRAKTSII UPRUTOI VOLHY NA OSTROM REBRE)

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In this paper we obtain an exact solution of the problem of diffraction of a transient plane elastic wave, with no resistance, propagated in three-dimensional space and striking against an edge in the form of a half-plane. The problem is solved by the method of functional-invariant solutions of V.I. Smirnov and S.L. Sobolev.

1. Consider the diffraction picture due to the motion of a plane elastic wave in space (x, y, z), which is occupied by a homogeneous isotropic elastic medium and in which a cut in the form of the half-plane $y = 0$, $x > 0$ has been made; the edge of the cut is fixed, i.e. the elastic displacement is equal to zero on this half-plane. The analogous diffraction problem for acoustic waves in a fluid was solved by Sobolev in [1; p.614].

It is known (see [1; pp.471-473]) that if there are no external forces, the displacement vector (u, v, w) can be written in the form

$$(u, v, w) = \operatorname{grad} \varphi + \operatorname{rot} \psi \tag{1.1}$$

where the scalar potential ϕ and the vector potential $\psi = (\psi_1, \psi_2, \psi_3)$ satisfy the equations

$$a^2 \frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}, \qquad b^2 \frac{\partial^2 \psi_i}{\partial t^2} = \frac{\partial^2 \psi_i}{\partial x^2} + \frac{\partial^2 \psi_i}{\partial y^2} + \frac{\partial^2 \psi_i}{\partial z^2} \quad (i = 1,2,3) \tag{1.2}$$

when $1/a$ and $1/b$ are the velocities of the longitudinal and transverse waves, respectively. On the fixed boundary (in our case on both sides of the half-plane $y = 0$, $x > 0$) the boundary conditions

$$u = 0, \qquad v = 0, \qquad w = 0 \tag{1.3}$$

hold.

It is required to investigate the diffraction picture resulting from the motion of the plane longitudinal wave

$$\varphi(t, x, y, z) = f(t - cz - c_1x + c_2y), \qquad \psi(t, x, y, z) = 0 \tag{1.4}$$

where $f(s) = 0$ for $s \leqslant 0$. The diffraction problem of the plane transverse wave is reducible to three diffraction problems as follows:

$$\begin{aligned} &1)\ \varphi = \psi_2 = \psi_3 = 0, \qquad \psi_1 = f_1(t - cz - c_1x + c_2y) \\ &2)\ \varphi = \psi_1 = \psi_3 = 0, \qquad \psi_2 = f_2(t - cz - c_1x + c_2y) \\ &3)\ \varphi = \psi_1 = \psi_2 = 0, \qquad \psi_3 = f_3(t - cz - c_1x + c_2y) \end{aligned}$$

Each of these three problems is solved by the same method as the diffraction problem of the wave (1.4). We shall, therefore, confine the discussion to the solution of the diffraction problem for waves of the form (1.4). We shall assume that $c > 0$ in (1.4), since the case $c < 0$ reduces to that of $c > 0$ if z is replaced by $-z$ and the case $c = 0$ is the plane problem* treated previously in [2].

At each instant of time t the front of the incident wave is a plane intersecting the axis Oz at the point $z = t/c$. This point is the vertex of the cone $t - cz > [(a^2 - c^2)(x^2 + y^2)]^{1/2}$ occupied by the diffracted waves (analogous to [1; p. 615]).

In the exterior of the cone there are only plane waves: the incident wave (1.4) and the two waves (the longitudinal and the transverse) reflected from the fixed boundary. At arbitrary $t = t_0$ the picture of the fronts of the wave in the plane section $z = z_0$ is the one given in Fig. 1 for $t > cz_0$ (if $t < cz_0$, then the front of the wave in the section considered has not yet reached the cut).

It is sufficient to solve the problem for the case

$$f(s) = 0 \ \text{ for } \ s \leqslant 0, \quad f(s) = s \ \text{ for } \ s \geqslant 0 \tag{1.5}$$

in (1.4), since an arbitrary wave of the form (1.4) can be obtained by superposition of such waves.

2. We introduce the notation

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* It can be verified that if we put $c = 0$ in the solution obtained in this paper and then differentiate the solution with respect to t (in order to pass from the initial conditions of the form (1.4) considered here, where $f(s) = s$ for $s > 0$, to the final conditions of [2], where $f(s) = 1$ for $s > 0$), then the solution of [2] is obtained.

$$a_1^2 = a^2 - c^2, \qquad b_1^2 = b^2 - c^2$$

$$u_0 = \frac{\partial \varphi}{\partial x}, \qquad v_0 = \frac{\partial \varphi}{\partial y}, \qquad w_0 = \frac{\partial \varphi}{\partial z} \tag{2.1}$$

$$u_1 = \frac{\partial \psi_3}{\partial y} - \frac{\partial \psi_2}{\partial z}, \qquad v_1 = \frac{\partial \psi_1}{\partial z} - \frac{\partial \psi_3}{\partial x}, \qquad w_1 = \frac{\partial \psi_2}{\partial x} - \frac{\partial \psi_1}{\partial y}$$

By (1.1) we have $(u_0, v_0, w_0) + (u_1, v_1, w_1) = (u, v, w)$. In analogy with [1; p. 615] we shall seek a solution depending only on the three variables x, y, $t_1 = t - cz$. From (1.2) it follows that each of the functions u_0, v_0, w_0 satisfies

$$a_1^2 \frac{\partial^2 u_0}{\partial t_1^2} = \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} \tag{2.2}$$

and that each of the functions u_1, v_1, w_1 satisfies

$$b_1^2 \frac{\partial^2 u_1}{\partial t_1^2} = \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} \tag{2.3}$$

From (2.1) we obtain

$$\frac{\partial u_0}{\partial y} = \frac{\partial v_0}{\partial x}, \qquad -c\frac{\partial u_0}{\partial t_1} = \frac{\partial w_0}{\partial x}, \qquad -c\frac{\partial v_0}{\partial t_1} = \frac{\partial w_0}{\partial y}, \qquad \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = c\frac{\partial w_1}{\partial t_1} \tag{2.4}$$

The boundary conditions (1.3) can be written as

$$u_0 + u_1 = 0, \qquad v_0 + v_1 = 0, \qquad w_0 + w_1 = 0 \quad \text{for } y = 0,\ x \geqslant 0 \tag{2.5}$$

Noting that (1.2) and (1.4) imply that $c^2 + c_1^2 + c_2^2 = a^2$ and putting $c_1 = k$, we obtain $c_2^2 = a_1^2 - k^2$. It now follows from (1.1), (1.4) and (1.5) that

$$u_0 = v_0 = w_0 = u_1 = w_1 = v_1 = 0 \qquad \text{for } t_1 - kx + \sqrt{a_1^2 - k^2}\,y < 0 \tag{2.6}$$

$$u_1 = v_1 = w_1 = 0, \quad u_0 = -k, \quad v_0 = \sqrt{a_1^2 - k^2}, \quad w_0 = -c \tag{2.7}$$

$$\text{for } t_1 - kx + \sqrt{a_1^2 - k^2}\,y > 0$$

For $t_1 > 0$ we have the picture shown in Fig. 1. Obviously, (2.6) holds in the regions MKx and M_1C_1AKx (before the wave front), and (2.7) holds in the region to the left of the curve $MKCA_1C_1M_1$ (that is, those places on the front of the incident wave not yet reached by the reflected and diffracted waves). The regions ACK and $AEDK$ contain the waves reflected from the plane boundary Ox in accordance with the boundary conditions (2.5).

In the usual way (see [2; p. 689]) we obtain

$$u_0 = p, \quad v_0 = q, \quad w_0 = r \text{ in the region } ACK$$

$$u_1 = -p, \quad v_1 = -q, \quad w_1 = -r \text{ in the region } AEDK$$

$$p = -\frac{2k\sqrt{a_1^2-k^2}\sqrt{b_1^2-k^2}}{F(k)}, \quad q = \frac{2(k^2+c^2)\sqrt{a_1^2-k^2}}{F(k)}. \tag{2.8}$$
$$r = -\frac{2c\sqrt{a_1^2-k^2}\sqrt{b_1^2-k^2}}{F(k)}, \quad F(k) = k^2+c^2+\sqrt{a_1^2-k^2}\sqrt{b_1^2-k^2}$$

An arbitrary function $\Phi(y)$ can be written as half the sum of the even function $\Phi^+(y) = \Phi(y) + \Phi(-y)$ and the odd function $\Phi^-(y)$ h $\Phi(y) = \Phi(-y)$. We shall denote by u_0^0, v_0^*, w_0^0, u_1^0, v_1^*, w_1^0 the even functions of y

$$u_0^\circ(t_1, x, y) = u_0(t_1, x, y) + u_0(t_1, x, -y)$$
$$v_0^*(t_1, x, y) = v_0(t_1, x, y) + v_0(t_1, x, -y)$$

etc. and by u_0^*, v_0^0, w_0^*, u_1^*, v_1^0, w_1^* the odd functions of y

$$u_0^*(t_1, x, y) = u_0(t_1, x, y) - u_0(t_1, x, -y)$$

etc. where $u_0, \ldots, w_1$ is a solution of the problem (2.1)-(2.7). It is obvious that $u_0 = 1/2(u_0^0 + u_0^*)$ etc. and that each of the systems of functions u_0^0, v_0^0, w_0^0, u_1^0, v_1^0, w_1^0 and u_0^*, v_0^*, w_0^*, u_1^*, v_1^*, w_1^* satisfies (2.2), (2.3), (2.4) and the boundary condition (2.5).

We shall first find the functions u_0^0, v_0^0, w_0^0, u_1^0, v_1^0, w_1^0. It follows from (2.6) and (2.7) that these functions are homogeneous of order zero in t_1, x, y for $t_1 < 0$. Therefore, the solution will be homogeneous for $t_1 > 0$ as well. According to [1; p. 514] such a solution can be sought in the form

$$u_0^\circ = \operatorname{Re} U_0(\theta_0), \quad v_0^\circ = \operatorname{Re} V_0(\theta_0), \quad w_0^\circ = \operatorname{Re} W_0(\theta_0)$$
$$u_1^\circ = \operatorname{Re} U_1(\theta_1), \quad v_1^\circ = \operatorname{Re} V_1(\theta_1), \quad w_1^\circ = \operatorname{Re} W_1(\theta_1) \tag{2.9}$$

where U_0, V_0, $\ldots$, W_1 are analytic functions of the complex variables

$$\theta_0 = \sigma + i\tau_0, \qquad \theta_1 = \sigma + i\tau_1$$
$$\sigma = \frac{t_1 x}{x^2+y^2} \qquad \tau_0 = \frac{y\sqrt{t_1^2 - a_1^2(x^2+y^2)}}{x^2+y^2}, \qquad \tau_1 = \frac{y\sqrt{t_1^2 - b_1^2(x^2+y^2)}}{x^2+y^2} \tag{2.10}$$

From (2.4) we obtain

$$\sqrt{a_1^2-\theta^2}\,U_0'(\theta) = \theta V_0'(\theta), \qquad cU_0'(\theta) = \theta W_0'(\theta)$$
$$cV_0'(\theta) = \sqrt{a_1^2-\theta^2}\,W_0'(\theta) \tag{2.11}$$
$$\theta U_1'(\theta) + \sqrt{b_1^2-\theta^2}\,V_1'(\theta) + cW_1'(\theta) = 0$$

The radicals are to be considered continuous for $\operatorname{Im}\theta \geqslant 0$ and positive for $-a_1 < \theta < a_1$.

Because the functions u_0^0, w_0^0, u_1^0, w_1^0 are even in y and the functions

$v_0{}^0$, $v_1{}^0$ are odd in y, it is sufficient to consider the functions $U_0(\theta)$, ..., $W_1(\theta)$ only in the upper half-plane. In the same way as in [2; Section 2] we obtain the following conditions which the functions $U_0(\theta)$, ..., $W_1(\theta)$ must satisfy on the real axis:

$$\operatorname{Im} U_0'(\theta) = 0, \quad \operatorname{Re} V_0(\theta) = 0, \quad \operatorname{Im} W_0'(\theta) = 0 \text{ for } -\infty < \theta < -a_1 \tag{2.12}$$

$$\operatorname{Im} U_1'(\theta) = 0, \quad \operatorname{Re} V_1(\theta) = 0, \quad \operatorname{Im} W_1'(\theta) = 0 \text{ for } -\infty < \theta < -b_1 \tag{2.13}$$

$$\operatorname{Re} U_0(\theta) = -2k, \quad \operatorname{Re} V_0(\theta) = 0, \quad \operatorname{Re} W_0(\theta) = -2c \text{ for } -a_1 < \theta < k \tag{2.14}$$

$$\operatorname{Re} U_1(\theta) = 0, \quad \operatorname{Re} V_1(\theta) = 0, \quad \operatorname{Re} W_1(\theta) = 0 \text{ for } -b_1 < \theta < k \tag{2.15}$$

$$\left.\begin{matrix} \operatorname{Re} U_0(\theta) = p, & \operatorname{Re} V_0(\theta) = q, & \operatorname{Re} W_0(\theta) = r \\ \operatorname{Re} U_1(\theta) = -p, & \operatorname{Re} V_1(\theta) = -q, & \operatorname{Re} W_1(\theta) = -r \end{matrix}\right\} \text{ for } k < \theta < a_1 \tag{2.16}$$

$$\left.\begin{matrix} \operatorname{Re}(U_0(\theta) + U_1(\theta)) = 0, & \operatorname{Re}(V_0(\theta) + V_1(\theta)) = 0 \\ \operatorname{Re}(W_0(\theta) + W_1(\theta)) = 0 & \end{matrix}\right\} \text{ for } a_1 < \theta < +\infty \tag{2.17}$$

3. We shall find functions $U_0(\theta)$, ..., $W_1(\theta)$, which are regular in the upper half-plane and satisfy (2.11) and also satisfy the boundary conditions (2.12)-(2.17) on the real axis. It follows from (2.12)-(2.17) that $\operatorname{Re}[V_0(\theta) + V_1(\theta)] = 0$ on the real axis. Therefore,*

$$V_0(\theta) + V_1(\theta) \equiv 0 \tag{3.1}$$

By means of (2.11) and (3.1) we express the functions U_0', V_0', U_1', V_1' in terms of W_0' and W_1'. Using also (2.17), we obtain

$$\operatorname{Re} F(\theta) W_0'(\theta) = 0 \text{ when } a_1 < \theta < +\infty \; (F(\theta) = \theta^2 + c^2 + \sqrt{a_1{}^2 - \theta^2}\sqrt{b_1{}^2 - \theta^2}) \tag{3.2}$$

From (2.12), (2.14) and (2.16) we obtain

$$\left.\begin{matrix} \operatorname{Im} W_0'(\theta) = 0 \quad \text{for } -\infty < \theta < -a_1 \\ \operatorname{Re} W_0'(\theta) = 0 \text{ for } -a_1 < \theta < k \text{ and } k < \theta < a_1 \end{matrix}\right\} \tag{3.3}$$

and for $\theta = k$ the function $W_0'(\theta)$ has a pole with principal part

$$W_0'(\theta) \sim \frac{i(r + 2c)}{\pi(\theta - k)} = \frac{i}{\pi(\theta - k)} \frac{2c(k^2 + c^2)}{F(k)} \tag{3.4}$$

From (3.2) and (3.3) we know arg $W_0'(\theta)$ is wholly on the real axis. In order to find $W_0'(\theta)$ we construct a function $F_1(\theta)$ such that

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* In view of the considerations at the beginning of Section 3 of [2], we shall seek the simplest particular solution of the problem (2.11)-(2.17), i.e. a solution with the minimum number of singularities on the real axis and at infinity.

$$\ln F_1(\theta) = \frac{1}{\pi}\int_{a_1}^{b_1} \frac{\arg F(t)}{\theta - t}\,dt = \frac{1}{\pi}\int_{a_1}^{b_1} \operatorname{arc\,tg} \frac{\sqrt{b_1^2 - t^2}\sqrt{t^2 - a_1^2}}{t^2 + c^2}\,\frac{dt}{t - \theta} \tag{3.5}$$

Then $F_1(\infty) = 1$; on the real axis $F_1(\theta) = 0$, except for the interval (a_1, b_1), where $\arg F_1(\theta) = -\arg F(\theta)$. It now follows from (3.2) and (3.3) that

$$\operatorname{Re} \frac{W_0'(\theta)\sqrt{\theta + a_1}}{F_1(\theta)} = 0$$

is wholly on the real axis, except at the point $\theta = k$, where there is a pole. Therefore,

$$\frac{W_0'(\theta)\sqrt{\theta + a_1}}{F_1(\theta)} = iA + \frac{iB}{\theta - k} \tag{3.6}$$

From (3.4) we obtain

$$B = \frac{2c(k^2 + c^2)\sqrt{k + a_1}}{\pi F(k) F_1(k)}$$

The constant A will be determined later. From (2.17) and (2.11) it follows that

$$\operatorname{Re} cU_1' = \operatorname{Re} \theta W_1' \qquad \text{for } \theta > a_1$$

From this and from (2.13), (2.15) and (2.16) it follows that

$$\operatorname{Re}\left[(cU_1'(\theta) - \theta W_1'(\theta))\sqrt{b_1 + \theta}\right] = 0 \text{ for } -\infty < \theta < +\infty \tag{3.7}$$

Equations (2.15), (2.16), (2.11) and (2.8) imply that $\operatorname{Re}(cU_1 - \theta W_1)$ is continuous at $\theta = k$; hence the function $cU_1' - \theta W_1'$ does not have a pole there. Using (3.7), we write

$$cU_1'(\theta) - \theta W_1'(\theta) = \frac{iD}{\sqrt{b_1 + \theta}} \tag{3.8}$$

From (2.1), (3.1), (3.6), (3.8) we find all six of the required functions

$$U_0'(\theta) = \frac{\theta}{c} W_0'(\theta) = \frac{i\theta F_1(\theta)}{c\sqrt{a_1 + \theta}}\left(A + \frac{B}{\theta - k}\right), \qquad U_1'(\theta) = \frac{\theta}{c} W_1'(\theta) + \frac{iD}{c\sqrt{b_1 + \theta}}$$

$$V_0'(\theta) = -V_1'(\theta) = \frac{i\sqrt{a_1 - \theta}F_1(\theta)}{c}\left(A + \frac{B}{\theta - k}\right) \tag{3.9}$$

$$W_1'(\theta) = \frac{i}{\theta^2 + c^2}\left[F_1(\theta)\sqrt{a_1 - \theta}\sqrt{b_1^2 - \theta^2}\left(A + \frac{B}{\theta - k}\right) - \frac{\theta D}{\sqrt{b_1 + \theta}}\right]$$

We find the constants A and D from the condition of regularity of the functions U_1' and W_1' at the point $\theta = ic$. to do so, we must put the expression in the square brackets in (3.9) equal to zero for $\theta = ic$, i.e.

$$F_1(ic)\sqrt{a_1 - ic}\sqrt{b_1 + ic}\sqrt{b_1^2 + c^2}\left(A + \frac{B}{ic - k}\right) - icD = 0$$

Putting the real part of this expression equal to zero, we find

$$A = \frac{B}{c^2 + k^2}\left(k - c\,\frac{\operatorname{Im}(F_1(ic)\sqrt{a_1 - ic}\sqrt{b_1 + ic})}{\operatorname{Re}(F_1(ic)\sqrt{a_1 - ic}\sqrt{b_1 + ic})}\right) \qquad (3.10)$$

and putting the imaginary part equal to zero, we find D. We show that the denominator in (3.10) does not vanish. Indeed, $0 < a_1 < b_1$ and $c > 0$ imply that

$$-\tfrac{1}{4}\pi < \arg(\sqrt{a_1 - ic}\sqrt{b_1 + ic}) < 0$$

and (3.5) implies that

$$\arg F_1(ic) = \operatorname{Im}\ln F_1(ic) < \frac{1}{\pi}\int_{a_1}^{b_1}\frac{\pi}{2}\operatorname{Im}\frac{1}{t - ic}\,dt < \frac{\pi}{4}$$

Hence

$$|\arg(F_1(ic)\sqrt{a_1 - ic}\sqrt{b_1 + ic})| < \tfrac{1}{4}\pi$$

and the denominator in (3.10) is not zero. Therefore, the functions U_0', ..., W_1' have been determined. Taking (2.14 and (2.15) into account, we find U_0, ..., W_1, and then from (2.9) we find u_0^0, ..., w_1^0. In the region BEA (see Fig. 1), where r_1 is imaginary, the functions u_1^0, v_1^0, w_1^0 are determined in the same way as the function ψ in [2; Section 2; paragraph 3].

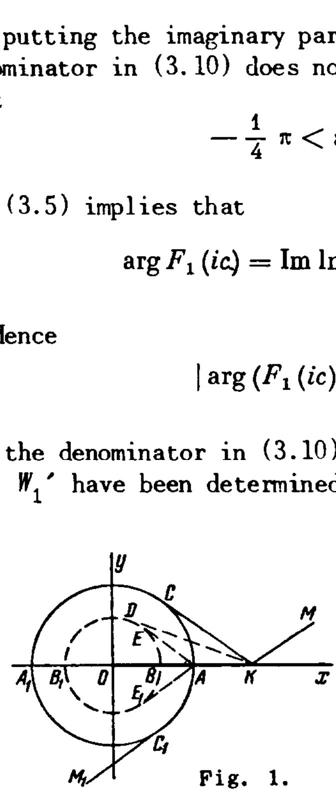

Fig. 1.

4. To find the functions u_0^*, ..., w_1^*, we write

$$u_0^* = \operatorname{Re}U_0^*(\theta_0), \ldots, w_1^* = \operatorname{Re}W_1^*(\theta_1)$$

in analogy with (2.9), where θ_0 and θ_1 are the same as in (2.10). The functions U_0^*, ..., W_1^* will satisfy the same relations (2.11), but instead of (2.12)-(2)14) they will satisfy the following boundary conditions:

$$\operatorname{Re}U_0^*(\theta) = 0, \quad \operatorname{Im}V_0^{*\prime}(\theta) = 0, \quad \operatorname{Re}W_0^*(\theta) = 0 \text{ for } -\infty < \theta < a_1 \qquad (4.1)$$

$$\operatorname{Re}U_1^*(\theta) = 0, \quad \operatorname{Im}V_1^{*\prime}(\theta) = 0, \quad \operatorname{Re}W_1^*(\theta) = 0 \text{ for } -\infty < \theta < -b_1 \qquad (4.2)$$

$$\operatorname{Re}U_0^*(\theta) = 0, \quad \operatorname{Re}V_0^*(\theta) = 2\sqrt{a_1^2 - k^2}$$
$$\operatorname{Re}W_0^*(\theta) = 0 \quad \text{for } -a_1 < \theta < k \qquad (4.3)$$

Conditions (2.15)-(2.17) remain in force for the functions U_0^*, ..., W_1^* as well. These conditions imply that

$$\operatorname{Re}(U_0^* + U_1^*) = 0, \qquad \operatorname{Re}(W_0^* + W_1^*) = 0$$

for $-\infty < \theta < \infty$.

Therefore, instead of a single equation (3.1) we obtain two:

$$U_0^* + U_1^* = 0, \qquad W_0^* + W_1^* = 0$$

The rest of the discussion proceeds as in Section 3 and we obtain

$$W_0^{*\prime}(\theta) = -W_1^{*\prime}(\theta) = \frac{c}{\theta} U_0^{*\prime}(\theta) = -\frac{c}{\theta} U_1^{*\prime}(\theta) = \frac{iE\sqrt{b_1-\theta}F_1(\theta)}{\theta-k} \tag{4.4}$$

$$V_0^{*\prime}(\theta) = \frac{iE\sqrt{b_1-\theta}\sqrt{a_1^2-\theta^2}F_1(\theta)}{c(\theta-k)}, \qquad V_1^{*\prime}(\theta) = \frac{iE(\theta^2+c^2)F_1(\theta)}{c(\theta-k)\sqrt{b_1+\theta}}$$

where the function $F_1(\theta)$ is the same as in (3.5) and

$$E = -\frac{2c\sqrt{a_1^2-k^2}\sqrt{b_1+k}}{\pi F(k)F_1(k)} = -\frac{4c\sqrt{a_1^2-k^2}\sqrt{b_1+k}F_1(-k)}{\pi(a_1^2+b_1^2+2c^2)} \tag{4.5}$$

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